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Local Copying of $d \times d$-dimensional Partially Entangled Pure States

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Abstract We examine the problem of copying a set of orthogonal, entangled partially (non-maximally) bipartite pure states with an entangled blank state under the restriction to local operations and classical communication (LOCC), and show a protocol for copying these states by LOCC. The necessary and sufficient condition for locally copying partially entangled pure states is then represented. As a result, we find that the problem of local copying these entangled states can be regarded to some extent as that of catalytic transformation between them by LOCC.

Keywords Local copying · Partially entanglement · Orthogonal

1 Introduction

Of all the epochmaking discoveries in quantum computation and quantum information theory, the no cloning theorem has an important position since it implies that nonorthogonal states cannot be cloned exactly [1], whereas orthogonal states can be perfectly copied in principle. If the given quantum states to be copied is entangled, however, we cannot necessarily copy these states perfectly in realistic situation, such as the case where Alice and Bob perhaps share an entangled state. In this case people have to restrict their operations to local operations and classical communication (LOCC). Thus, it is interesting from both viewpoints of entanglement theory and cloning theory to investigate copying entangled states. This problem is referred to local copying of entangled states [2]. A local cloning machine has several potential applications. In particular, in secret sharing [3], a boss requires sending the states $\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, $\frac{1}{\sqrt{2}}(|00\rangle \pm i|11\rangle)$ to two subordinates. Note that a local attack is the realistic one for secret sharing, if the attacker can locally copying these states to be sent.

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Fruitful results on copying have also been obtained under the restriction to LOCC. For example, Ghosh et al. considered the problem of copying orthogonal Bell states under LOCC in [2], and found that some sets of orthogonal, maximally entangled states can be copied by LOCC and with a maximally entangled blank state. Ghosh et al. [2] showed, however, that perfect local copying cannot be accomplished without a blank state containing one ebit of entanglement. Anselmi [4] studied in detail the LOCC copying problem where both the blank state and all of the states to be copied are maximally entangled pure states, and obtained a necessary and sufficient condition for LOCC copying under these conditions. Owari [5] focused on the non-locality concerning local copying and local discrimination for a set of orthogonal maximally entangled states in prime dimensional Hilbert space. As a result, the authors shown that local copying is more difficult than local discrimination. Since a maximally entangled state cannot be kept a long time in realistic situation, it is more useful to consider the problem of local copying partially entangled states. In [6] the authors provided a complete characterization of the partially states that can be cloned. They considered that the set of clonable states is very restrictive—they must be locally distinguishable with a round of simultaneous communication, and they must have the same entanglement. However, a set of locally distinguishable states does not have to be locally copied according to [5]. This is obviously contradictory. So we will reconsider the condition for local copying partially entangled states in this paper. We analyze in detail the condition for locally copying n orthogonal entangled states with d dimensional subsystems with the blank state being also chosen as a non-maximally entangled state whose subsystems are d dimensional. In terms of Kraus operators, we take the entire copying operations consisting of just unitary operations and local positive operator-valued measure (POVM), which are carried out by each party. And then we provide a protocol for LOCC copying of $d \times d$-dimensional non-maximally entangled states with an entangled blank state utilizing the convenient form of these operators. We divide the copying protocol into two steps. First, Alice and Bob perform unitary operations to copy some maximally entanglement states $|\psi_{\max}^j\rangle$ on the system $\mathcal{H}_3 \otimes \mathcal{H}_4$. Secondly, a POVM is applied to remove the entanglement from these maximally entanglement states. When a POVM is applied by a single party, say Bob, and Alice only applies unitaries in response to it, this will always remove same amount of the entanglement from the blank state $|a^{34}\rangle$ onto ones to be copied.

2 The Problem of Local Copying Partially Entangled States

First of all, we focus on the perfect copying of bipartite states under the following restrictions:

1. Our operations are restricted to LOCC.
2. A set of entangled bipartite partially states, say $\{|\psi_j\rangle\}_{j=1}^n$, to be copied is orthogonal. This implies that, without the LOCC restriction, the states could be perfectly copied.
3. $|\psi_j^{12}\rangle$ $(j = 1, 2, \dots, n)$ has the same entanglement.

On the other hand, Alice and Bob each have two d-dimensional quantum systems denoted by $\mathcal{H}_{\mathcal{A}} = \mathcal{H}_1 \otimes \mathcal{H}_3$ and $\mathcal{H}_{\mathcal{B}} = \mathcal{H}_2 \otimes \mathcal{H}_4$, respectively. Particles 1 and 2 are initially prepared in one of these states $|\psi_j^{12}\rangle$ while particles 3 and 4 are initially prepared in the blank state $|a^{34}\rangle$. Alice and Bob aim to perform the transformation

$$|\psi_j^{12}\rangle \otimes |a^{34}\rangle \to |\psi_j^{12}\rangle \otimes |\psi_j^{34}\rangle \tag{1}$$

by LOCC. Here, the superscripts indicate the particles that have been prepared in each state. According to Kraus representation [7], Alice and Bob's performance can be described using the quantum operations

$$\sum_k (A_k^{13} \otimes B_k^{24})(|\psi_j^{12}\rangle|a^{34}\rangle\langle\psi_j^{12}|\langle a^{34}|)(A_k^{13} \otimes B_k^{24})^\dagger = |\psi_j^{12}\rangle|\psi_j^{34}\rangle\langle\psi_j^{12}|\langle\psi_j^{34}| \tag{2}$$

where the operator A_k^{13} and B_k^{24}, acting on $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ respectively, must be unitary or measurement operators that satisfy $\sum_k (A_k^{13} \otimes B_k^{24})^\dagger (A_k^{13} \otimes B_k^{24}) \leq I$.

Since LOCC operations do not increase the entanglement of whole states, $\{|\psi_j\rangle\}_{j=1}^n$ cannot be copied by LOCC without any entanglement resource. Thus, we also assume that Alice and Bob share a blank entangled state $|a^{34}\rangle$. Due to the limitations on the LOCC manipulation of entanglement, it is a non-trivial matter to determine the set of blank states which enable one to copy by LOCC. The conditions under which this is possible can be obtained using Nielsen's theorem [8]. Let $\lambda_{\psi_0} = (\lambda_\psi^1, \lambda_\psi^2, \ldots, \lambda_\psi^d)$ be the Schmidt vector of $|\psi_0\rangle$ with decreasing ordered. Nielsen's theorem implies that

$$|\psi_0^{12}\rangle \otimes |a^{34}\rangle \stackrel{\text{LOCC}}{\longrightarrow} |\psi_0^{12}\rangle \otimes |\psi_0^{34}\rangle \quad \Longleftrightarrow \quad \lambda_{\psi_0} \otimes \lambda_a \otimes \prec \lambda_{\psi_0} \otimes \lambda_{\psi_0} \tag{3}$$

where $\prec$ denotes the majorization relation. Clearly, this copying transformation will be possible if the transformation $|\psi_0^{12}\rangle \otimes |a^{34}\rangle \rightarrow |\psi_0^{12}\rangle \otimes |\psi_0^{34}\rangle$ is possible by LOCC. Thus, we can choose the blank state $|a\rangle$ in terms of $|\psi_0\rangle$ serve as "catalysts" for pure, bipartite entanglement transformations [9]. Therefore, there are two possible options of the blank state:

(1) $|a\rangle \stackrel{\text{LOCC}}{\longrightarrow} |\psi_0\rangle$. This case means that $\lambda_a \prec \lambda_{\psi_0}$.

(2) $|a\rangle \nrightarrow |\psi_0\rangle$ and $|\psi_0\rangle|a\rangle \stackrel{\text{LOCC}}{\longrightarrow} |\psi_0\rangle|\psi_0\rangle$. In this case, $|\psi_0\rangle$ serves as a catalyst for the transformation $|a\rangle|$ into $|\psi_0\rangle|$. Let $l_u(|\psi_0\rangle)$ and $g_u(|\psi_0\rangle)$ denote $l_u(|\psi_0\rangle) = \min\{\frac{\lambda_\psi^{i+1}}{\lambda_\psi^i},\ 1 \leq i \leq d\}$ and $g_u(|\psi_0\rangle) = \frac{\lambda_\psi^d}{\lambda_\psi^1}$, respectively [10, 11]. Thus we can take a blank state $|a\rangle$ such that $l_u(|a\rangle) > g_u(|\psi_0\rangle)$. More discussions about the mathematical structures of quantum catalysis see [10, 11]. Such sets will be the focus of our attention for the remainder of this paper.

3 Condition for Local Copying Partially Entangled Pure States

To begin with, let $\{|e_i\rangle\}_{i=1}^d$ be an orthonormal basis for the d-dimensional Hilbert space $\mathcal{H}$. $|\psi_{\max}\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |e_i\rangle|e_i\rangle$ consequently is a maximally entangled state in $\mathcal{H}^{\otimes 2}$. If linear operators A and B can be expressed in the basis as the matrix $A = (a_{ij})_{d\times d}$ and $B = (b_{kl})_{d\times d}$ respectively, then

$$(A \otimes B)|\psi_{\max}\rangle = (I \otimes BA^{\mathrm{T}})|\psi_{\max}\rangle = (AB^{\mathrm{T}} \otimes I)|\psi_{\max}\rangle \tag{4}$$

where A^{T} is the transposition of A. In particular,

$$(A \otimes I)|\psi_{\max}\rangle = (I \otimes A^{\mathrm{T}})|\psi_{\max}\rangle \tag{5}$$

As a result, let A, B and C be linear operators in Hilbert space $\mathcal{H}$. Then the following equation holds:

$$(A \otimes BC)|\psi_{\max}\rangle = (AC^{\mathrm{T}} \otimes B)|\psi_{\max}\rangle \tag{6}$$

Let $|\psi_0^{12}\rangle = (M^1 \otimes I^2)|\psi_{\max}^{12}\rangle$. According to the Schmidt decomposition, M can be rewritten as $M = \mathrm{diag}(\lambda_\psi^1\ \lambda_\psi^2 \cdots \lambda_\psi^d)$. Recall the third restriction mentioned above. Since $|\psi_j^{12}\rangle$ $(j = 1, 2, \ldots, n)$ has the same entanglement, they are equivalent under local unitary transformations. On the other hand, there exist unitary operators V_j^1 on $\mathcal{H}_1$ and W_j^2 on $\mathcal{H}_2$ such that $|\psi_j^{12}\rangle = (M^1 \otimes I^2)(V_j^1 \otimes W_j^2)|\psi_{\max}\rangle$. By (4), $|\psi_j^{12}\rangle$ can be rewritten as $(V_j^1(W_j^1)^{\mathrm{T}} \otimes I^2)|\psi_0^{12}\rangle$. Therefore, it is enough to consider this case that $|\psi_j^{12}\rangle = (U_j^1 \otimes I^2)|\psi_0^{12}\rangle$, where U_j $(j = 1, 2, \ldots, n)$ is unitary.

Returning to the problem of locally copying the n states $|\psi_j^{12}\rangle$, this implies that there exists some global Kraus operators $A_k^{13} \otimes B_k^{24}$ such that $A_k^{13} \otimes B_k^{24}(|\psi_j^{12}\rangle \otimes |a^{34}\rangle) = p_{kj}|\psi_j^{12}\rangle \otimes |\psi_j^{12}\rangle$ (seeing (2)), where $\sum_k |p_{kj}|^2 = 1$. For the sake of convenience, we may drop the index k. In terms of these discussions, the copying transformation will have the form

$$\begin{aligned}(A^{13} \otimes B^{24})|\psi_j^{12}\rangle|a^{34}\rangle &= (A^{13}(U_j^1 \otimes I^3) \otimes B^{24}(M_2^{\mathrm{T}} \otimes N_4^{\mathrm{T}}))|\psi_{\max}^{12}\rangle|\psi_{\max}^{34}\rangle \\ &= p_j(U_j^1 \otimes U_j^3 \otimes M_2^{\mathrm{T}} \otimes M_4^{\mathrm{T}})|\psi_{\max}^{12}\rangle|\psi_{\max}^{34}\rangle \end{aligned} \tag{7}$$

Now we consider a copy protocol including two step to locally copy the n states $|\psi_j\rangle$. First, Alice and Bob copy $(U_j^3 \otimes I^4)|\psi_{\max}^{34}\rangle$ on the system $\mathcal{H}_3 \otimes \mathcal{H}_4$ by LOCC. So as to achieve this aim, Alice applies a unitary operator A_{13} on her system $\mathcal{H}_{\mathcal{A}}$ in order to ensure that the following equation holds:

$$A_{13}(U_j^1 \otimes I^3)A_{13}^* = (U_j^1 \otimes U_j^3) \tag{8}$$

And then Bob consequently applies unitary operator A_{24}^* on $\mathcal{H}_{\mathcal{B}}$, where A^* is complex conjugate of A. It is reason for it that the left hand of (7) becomes $(A_{13}(U_j^1 \otimes I_3)A_{24}^* \otimes (M_2^{\mathrm{T}} \otimes N_4^{\mathrm{T}}))|\psi_{\max}^{12}\rangle|\psi_{\max}^{34}\rangle = (U_j^1 \otimes U_j^3) \otimes (M_2^{\mathrm{T}} \otimes N_4^{\mathrm{T}}))|\psi_{\max}^{12}\rangle|\psi_{\max}^{34}\rangle$.

According to [4], there exists an operator A such that (8) can hold for some specific operators U_j $(j = 1, 2, \ldots, n)$. However, it is not easy to find it. In the rest of this paper, in order to resolve (8), $d = \dim\mathcal{H}$ is restricted to being prime. Inspired by the ideas in [5], suppose that $U_j = \sum_k \omega^{n_j k}|e_k\rangle\langle e_k|$ $(n_j \in \mathbb{N})$, where ω is the d-th root of unity. We make a survey of A expressed as

$$A = \sum_{l,m} a_{lm}|e_{l-m \bmod d}\rangle\langle e_l| \otimes |e_m\rangle\langle e_m|, \quad \text{with } a_{lm} \in \mathbb{C} \text{ and } |a_{lm}|^2 = 1 \tag{9}$$

It is not difficult to verify that it is a solution of (8). Obviously, A is unitary. Moreover, $A(U_j \otimes I)A^* = \sum_{l,m,k,l',m'} a_{lm}a_{l'm'}^*\omega^{n_j k}|e_{l-m \bmod d}\rangle\langle e_l|e_k\rangle\langle e_k|e_{l'}\rangle\langle e_{l'-m' \bmod d}| \otimes |e_m\rangle\langle e_m|e_{m'}\rangle\langle e_{m'}| = \sum_{l,m} a_{lm}a_{lm}^*\omega^{n_j l}|e_{l-m \bmod d}\rangle\langle e_{l-m \bmod d}| \otimes |e_m\rangle\langle e_m|$. Set $l - m \equiv s \bmod d$, it can be rewritten as $\sum_{m,s} \omega^{n_j(m+s \bmod d)}|e_s\rangle\langle e_s| \otimes |e_m\rangle\langle e_m| = U_j \otimes U_j$. After Alice and Bob preforming that mentioned above, the left hand of (7) becomes $(U_j^1 \otimes M_2^{\mathrm{T}} \otimes U_j^3 \otimes N_4^{\mathrm{T}})|\psi_{\max}\rangle|\psi_{\max}\rangle$.

Secondly, Bob performs a POVM on his particles and Alice only applies unitary in response to it in order to obtain the right hand of (7). It is reasonable because two-way-communications strategy of entanglement manipulation is equivalent to a strategy involving only a single measurement by Bob followed by one-way communications of his result to Alice [12]. This implies that the following equation must be satisfied:

$$(A_l^{13} \otimes B_l^{24})(U_j^1 \otimes M^2 \otimes U_j^3 \otimes N^4) = q_l(U_j^1 \otimes M^2 \otimes U_j^3 \otimes M^4) \tag{10}$$

In other words, Alice and Bob perform to transform $|a^{34}\rangle = (I^3 \otimes N^4)|\psi_{\max}^{34}\rangle$ into $|\psi_0^{34}\rangle = (I^3 \otimes M^4)|\psi_{\max}^{34}\rangle$ by LOCC. For arbitrary two Hermite operators A and B, assume λ_A and λ_B are their vector of eigenvalue, respectively. If we define $A \prec B \Leftrightarrow \lambda_A \prec \lambda_B$, then $|a\rangle \xrightarrow{\text{LOCC}} |\psi_0\rangle \Leftrightarrow N^\dagger N \prec M^\dagger M$. Now we consider the two options of blank state $|a\rangle$.

(1) $|a\rangle \xrightarrow{\text{LOCC}} |\psi_0\rangle$. In this case, we have $N^\dagger N \prec M^\dagger M$. This implies that there exist a probability distribution q_l and unitary operators S_l such that $N^\dagger N = \sum_l q_l S_l (M^\dagger M) S_l^\dagger$ according to Uhlmann Theorem [8]. Thus, we can choose a set of operators M_l such that $M_l \sqrt{N^\dagger N} = \sqrt{q_l M^\dagger M} S_l^\dagger$. Next we verify M_l is a POVM. Without loss of generality, assume $\sqrt{N^\dagger N}$ is reversible. Therefore, M_l can be denoted by $M_l = \sqrt{q_l M^\dagger M} S_l^\dagger (N^\dagger N)^{-\frac{1}{2}}$. $\sum_l M^\dagger M_l = \sum_l (N^\dagger N)^{-\frac{1}{2}} (\sqrt{q_l M^\dagger M} S_l^\dagger)^\dagger \sqrt{q_l M^\dagger M} S_l^\dagger (N^\dagger N)^{-\frac{1}{2}} = (N^\dagger N)^{-\frac{1}{2}} N^\dagger N (N^\dagger N)^{-\frac{1}{2}} = I$. Let Bob apply measurement operators $B_l^{24} = I^2 \otimes M_l^4$ and Alice responses it using $A^{13} = I^1 \otimes S_l^3$ if Bob gets the result l.

(2) $|a\rangle \nrightarrow |\psi_0\rangle$ and $|\psi_0\rangle|a\rangle \xrightarrow{\text{LOCC}} |\psi_0\rangle|\psi_0\rangle$. This case implies $M^\dagger M \otimes N^\dagger N \prec M^\dagger M \otimes M^\dagger M$. Similar to (1), there exist a probability distribution q'_m and unitary operators T_m such that $M^\dagger M \otimes N^\dagger N = \sum_m q'_m T_m (M^\dagger M \otimes M^\dagger M) T_m^\dagger$. Thus Bob applies measurement operators N_m such that satisfy the condition $N_m \sqrt{M^\dagger M \otimes N^\dagger N} = \sqrt{q'_m M^\dagger M \otimes M^\dagger M} T_m^\dagger$, and Alice responses it using $A^{13} = I^1 \otimes T_m^3$ if Bob gets the result m.

In a word, if we choose blank state $|a^{34}\rangle$ such that $|\psi_0^{12}\rangle|a^{34}\rangle \xrightarrow{\text{LOCC}} |\psi_0^{12}\rangle|\psi_0^{34}\rangle$, then

Theorem 3.1 *A set of partially entangled pure states $|\psi_j\rangle = (U_j \otimes I)|\psi_0\rangle$ in prime dimensional quantum system can be locally copied with a blank state $|a\rangle$ if there exist a basis $\{|e_i\rangle\}_{i=1}^d$ and a set of integers $\{n_j\}_{j=1}^n$ such that the unitary U_j can be written as $U_j = \sum_k \omega^{n_j k} |e_k\rangle\langle e_k|$, where ω is the d-th root of unity.*

It is very interesting to find whether exists another set of partially entangled pure states which can be locally copied. It is very surprised for us to conclude that

Theorem 3.2 *A set of partially entangled pure states $|\psi_j\rangle = (U_j \otimes I)|\psi_0\rangle$ in prime dimensional quantum system can be locally copied if and only if $U_j = \sum_k \omega^{n_j k} |e_k\rangle\langle e_k|$, where ω is the d-th root of unity.*

Remark This Theorem manifests the category of bipartite entangled pure states which can be locally copied. Therefore, we find necessary and sufficient condition for local copying partially entangled pure states.

Proof ($\Leftarrow$) We have already proven that $|\psi_j\rangle = (U_j \otimes I)|\psi_0\rangle$ can be copied by LOCC in Theorem 3.1.

($\Rightarrow$) Similar to argument above, if a set of bipartite entangled pure states can be locally copied, then it means that, for two given local Kraus operators $\{A_k\}$, $\{B_k\}$ and a set of complex numbers p_k which meet condition $\sum_k |p_k|^2 = 1$, the solution of equation which is depicted as follows is not unique.

$$A_k (X \otimes I) B_k^{\mathrm{T}} = p_k (X \otimes X) \tag{11}$$

where X is an unknown diagonalized matrix. If we drop the index k, (11) can be rewritten as

$$A (X \otimes I) B^{\mathrm{T}} = p (X \otimes X) \tag{12}$$

Notice that there exist unitaries U_A, V_A, U_B and V_B such that $A = U_A\Sigma_A V_A$ and $B = U_B\Sigma_B V_B$ according to their singular value decomposition, where Σ_A and Σ_B is diagonalized matrix of $\sqrt{A^\dagger A}$ and $\sqrt{B^\dagger B}$, respectively. Substituting $U_A\Sigma_A V_A$ and $U_B\Sigma_B V_B$ for A and B in (12) respectively, we obtain

$$U_A\Sigma_A V_A(X\otimes I)V_B^{\mathrm{T}}\Sigma_B U_B^{\mathrm{T}} = p(X\otimes X) \tag{13}$$

Since $V_A(X\otimes I)V_B^{\mathrm{T}}$ and $X\otimes I$ have the same singular values, (13) implies that $\Sigma_A X\otimes I\Sigma_B$ and $p(X\otimes X)$ have the same singular values. In this case of X is unitary, (13) becomes the problem of locally coping maximally entangled pure states, which has been resolved in [5]. So we assume that X is an unknown positive semidefinite diagonalized matrix (this corresponds to states to be locally copied are partially entangled). However, the singular values of X is equivalent to its eigenvalues. Hence, (13) means that $\Sigma_A(X\otimes I)\Sigma_B$ and $p(X\otimes X)$ have the same eigenvalues. Without loss of generality, we take $d = 2$ as an example to resolve (13). The other cases can be deduced similarly. Suppose $\Sigma_A = \mathrm{diag}(a_1\ a_2\ a_3\ a_4)$, $\Sigma_B = \mathrm{diag}(b_1\ b_2\ b_3\ b_4)$ and $X = \mathrm{diag}(x_1\ x_2)$. Here we permit some $a_i(b_i)$ $(i = 1,2,3,4)$ equal zero, on the other hand, $A(B)$ may is non-full rank. According to argument above, it implies that $\{a_1b_1x_1,\ a_2b_2x_1,\ a_3b_3x_2,\ a_4b_4x_2\}$ equals $\{px_1^2,\ px_1x_2,\ px_1x_2,\ px_2^2\}$. Considering 24 possible options, we find that x_1 or x_2 can only be expressed as a function of variable a_i, b_i and p. This means that (12) has unique non-zero solution unless $a_ib_i = p$ $(i = 1,2,3,4)$ holds. Thus, it is sufficient to consider this case $a_ib_i = p$ $(i = 1,2,3,4)$. In this case, we obtain $X = 0$, I and $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. This means that these states can be written as $|\psi_j\rangle = (U_j\otimes I)|\psi_0\rangle$.

We would also like to tackle the question of locally catalytic copying. In this situation, some other entangled states which can be used in the protocol, but must be returned unchanged at the end of the copying process:

$$|\psi_j\rangle|a\rangle|b\rangle \xrightarrow{\mathrm{LOCC}} |\psi_j\rangle|\psi_j\rangle|b\rangle \tag{14}$$

where $|b\rangle$ acts as a catalyst, in much the same way as conversion between some states can only occur with the help of a catalyst. This means the following condition must be satisfied

$$A(U_jM\otimes N\otimes N')B^{\mathrm{T}} = p_j(U_jM\otimes U_jM\otimes N') \tag{15}$$

As argument above, we can split (15) into the following two equations:

$$A(U_j\otimes I\otimes I)A^\dagger = (U_j\otimes U_j\otimes I) \tag{16}$$

$$A(M\otimes N\otimes N')B^{\mathrm{T}} = p_j(M\otimes M\otimes N') \tag{17}$$

It is easy to see that the process of locally catalytic copying depend completely on the solution in (17). Hence, a catalyst cannot provide any enhancement in the local copying process. □

4 Application

We apply our local copying protocol to a set of mixed state $\rho_i = \sum_j^d p_{ij}|\psi_j\rangle\langle\psi_j|$, where p_{ij} denotes a probability distribution on a set $\{1,2,\ldots,d\}$ and $|\psi_j\rangle = (U_j\otimes I)|\psi_0\rangle$. Following the definition presented in [13], we define local "broadcasting" for a set of mixed state $\{\rho_i\}_{i=1}^n$ as follows. System $\mathcal{H}_\mathcal{A}$ is secretly prepared in one state from a set $\mathcal{A} = \{\rho_1, \rho_2, \ldots, \rho_n\}$ of n quantum states. System $\mathcal{H}_\mathcal{B}$ receives the unknown state with a blank quantum state Σ. The initial state of the composite system $\mathcal{H}_{\mathcal{AB}}$ is the product state $\rho_i \otimes \Sigma$ $(i = 1, 2, \ldots, n)$. If there is a local physical process $\varepsilon_{\text{LOCC}}$, consistent with the laws of quantum theory, that leads to an evolution $\varepsilon_{\text{LOCC}}(\rho_i \otimes \Sigma)$ such that

$$\text{tr}_A(\varepsilon_{\text{LOCC}}(\rho_i \otimes \Sigma)) = \text{tr}_B(\varepsilon_{\text{LOCC}}(\rho_i \otimes \Sigma)) = \rho_i \tag{18}$$

Here tr_A and tr_B denote partial traces over $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$, then the set $\mathcal{A}$ can be locally broadcast.

Remark Obviously, the word local copying for this strong form of local broadcasting.

By Theorem 3.1, mixed states $\rho_i = \sum_j^d p_{ij} |\psi_j\rangle\langle\psi_j|$ can be locally broadcast. However, they cannot be discriminated by LOCC according the following:

Proposition 4.1 [14] *The number of states n that can be discriminated perfectly by LOCC is bounded by*

$$n \leq \frac{d}{\frac{1}{n}\sum_{i=1}^n 2^{(E_R(\rho_i)+S(\rho_i))}} \tag{19}$$

where $E_R(\rho)$ denotes the relative entropy of entanglement; $S(\rho)$ is the von Neumann entropy.

A set of orthogonal partially entangled states $\{|\psi_j\rangle\}$ in a prime dimensional system is locally copiable, if and only if it can be expressed as $|\psi_j\rangle = (U_j \otimes I)|\psi_0\rangle$ with $U_j = \sum_k \omega^{n_j k} |k\rangle\langle k|$. Using this result, we can prove the fact that the maximal size of locally copiable sets is equal to the dimension of the local space as well as the maximal size of local distinguishable sets. Similar to the case of maximally entangled states (see [5]), we also show that if such a set is locally copiable, then locally distinguishable by one-way communication. Thus, in this case, local copying is strictly more difficult than one-way local discrimination. The relationship of local copiability and distinguishability for pure states is summarized in Fig. 1.

Recalling our protocol mentioned in Sect. 3, generally speaking, it is difficult to be locally copy arbitrary bipartite entangled mixed states since it is still a challenging task to find the condition for a bipartite entangled mixed state ρ transform into another σ by LOCC. However, we can find a set of entangled mixed states is locally broadcasted but locally distinguishable. Although the LOCC broadcast is not equal to LOCC copying for entangled mixed states, this perhaps implies that non-locality of pure states differ from that of mixed

Fig. 1

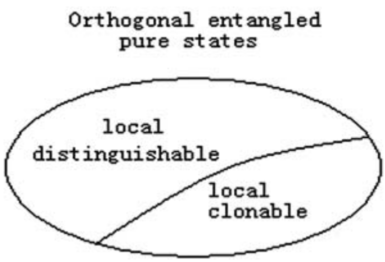
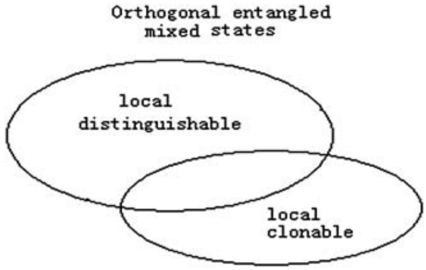


Fig. 2



ones if we focus on non-locality of a set of states by means of local copying and local discrimination. Therefore, our result may be useful to discover the non-locality problems of mixed states.

Conjecture The interrelationship of local copying and local discrimination for mixed states is depicted as Fig. 2.

In summary, we consider the problem of LOCC copying of partially entangled bi-partite states with an entangled blank state. And then a necessary and sufficient condition of local copying partially entangled states is shown. this results can be regarded as an application of the well known phenomenon of entanglement catalysis. As a result, we find that the blank state can be selected as not only a maximally entangled state but also a partially one according to [10, 11]. Thus, our protocol of LOCC copying is more useful in realistic situation. In order to solve (7), we restrict the states to locally copy in prime-dimensional quantum systems. However, the validity of Theorem 3.2 for arbitrary dimensional still remains as an open problem. We expect that the method to locally copy mixed states can be discovered, too.

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